General Relativity $\boxed{\text{Week 12}}$

We want to prove the local well-posedness for the initial value
in time
problem

(I) $$\begin{cases} \partial_\alpha \left(G^{\alpha\beta}(\psi)\,\partial_\beta \psi \right) = N(\psi, \partial\psi) & \text{on } \mathbb{R}^{1+n} \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1 \end{cases}, \quad \boxed{N(0,0)=0}$$

when $(\psi_0, \psi_1) \in H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n)$ for $k > 2n+2$.

We will measure the size of solutions using the higher order energy norm:

$$\mathcal{E}^{(k)}[f](\tau) = \sum_{|a| \le k} \int_{t=\tau} |\partial^a f|^2 \, dx$$

$$\partial^a = \partial_0^{a_0} \partial_1^{a_1} \cdots \partial_n^{a_n}$$

Note: $\mathcal{E}^{(k)}[f](\tau) = \sum_{i=0}^{k} \| \partial_t^i f|_{t=\tau} \|^2_{H^{k-i}(\mathbb{R}^n)}$

If ψ solves (I): We can express $\mathcal{E}^{(k)}[\psi](0)$ in terms of the initial data: $\| \psi|_{t=0} \|^2_{H^k} = \|\psi_0\|^2_{H^k}$, $\| \partial_t \psi|_{t=0} \|^2_{H^{k-1}} = \|\psi_1\|^2_{H^{k-1}}$

and, using the equation

$i, j = 1, 2, \ldots, n$

$$\partial_t^2 \psi|_{t=0} = -\frac{1}{G^{00}(\psi)} \left(2G^{0i}(\psi)\partial_i\partial_0\psi + G^{ij}(\psi)\,\partial_i\partial_j\psi + \tilde{N}(\psi, \partial\psi) \right)\Big|_{t=0}$$

Can be expressed in terms of (space derivatives of) ψ_0 and ψ_1.

Inductively: We can do the same for $\partial_t^{2+m} \psi|_{t=0}$

Using Sobolev embedding (and the fact that $k > 2n+2$):

We can bound $\mathcal{E}^{(k)}[\psi](0) \leq C(\|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}})$, where $C(\cdot)$ can be expressed in terms of the precise form of the functions $G^{\alpha\beta}$, N, etc.

When $\|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}} \lesssim 1$: We have $\mathcal{E}^{(k)}[\psi](0) \sim \|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}$

Theorem: Consider the initial value problem ①, where $|G^{\alpha\beta} - \eta^{\alpha\beta}| < \frac{1}{10 \cdot n}$ and $k > 2n+2$. set

$$M = \mathcal{E}^{(k)}[\psi](0)$$

Then $\exists\, T = T(M) > 0$ and a unique solution ψ on $[0,T] \times \mathbb{R}^n$ satisfying

$$\sup_{\tau \in [0,T]} \mathcal{E}^{(k)}[\psi](\tau) \leq C \cdot M$$

where $C > 1$ is independent of M, (ψ_0, ψ_1), T.

Moreover, ψ depends continuously on (ψ_0, ψ_1) and $T(M) \to \infty$ as $M \to 0$.

Remark: Since $\|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^k} \leq \mathcal{E}^{(k)}[\psi](0) \leq C\left(\|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}\right)$,

we could equivalently say that T is a function of $\|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}$.

Proof:

Existence: We will use a Picard iteration scheme:

Let us consider the sequence of functions $\psi^{(m)} : \mathbb{R}^{n+1} \to \mathbb{R}$ defined recursively by solving the linear problems:

- For $m = 0$:
$$\begin{cases} \partial_\alpha\left(G^{\alpha\beta}(0)\, \partial_\beta \psi^{(0)}\right) = 0 \\ \psi^{(0)}\big|_{t=0} = \psi_0, \quad \partial_t \psi^{(0)}\big|_{t=0} = \psi_1 \end{cases}$$

- For $m \geq 1$:
$$\begin{cases} \partial_\alpha\left(G^{\alpha\beta}(\psi^{(m-1)})\, \partial_\beta \psi^{(m)}\right) = N(\psi^{(m-1)}, \partial\psi^{(m-1)}) \\ \psi^{(m)}\big|_{t=0} = \psi_0, \quad \partial_t \psi^{(m)}\big|_{t=0} = \psi_1 \end{cases}$$

We will show that, if $T = T(M)$ is small enough in terms of M (to be determined precisely later) then, for any $m \geq 0$:

① $\begin{cases} \sup_{\tau\in[0,T]} \mathcal{E}^{(k)}[\psi^{(m)}](\tau) \leq C_0 \cdot M & (C_0: \text{Uniform constant}) \\ \sup_{\tau\in[0,T]} \mathcal{E}^{(1)}[\psi^{(m)}-\psi^{(m-1)}](\tau) \leq \frac{1}{2} \sup_{\tau\in[0,T]} \mathcal{E}^{(1)}[\psi^{(m)}-\psi^{(m-1)}](\tau) \end{cases}$

If ① holds, then it is easy to see (through "soft" functional-analytic arguments) that $\psi^{(m)}$ converges to a solution of Ⓘ on $[0,T]\times\mathbb{R}^n$:

① immediately implies that the sequence $\{\psi^{(m)}\}_{m=0}^{\infty}$ is <u>Cauchy</u> in the space $L_t^\infty \mathcal{E}^{(1)}$ (where $\mathcal{E}^{(k)}$ is the space of functions equipped with the k-th order energy norm defined before) and <u>uniformly bounded</u> in the higher order space $L_t^\infty \mathcal{E}^{(k)}$.

As a result: $\exists$ limiting function ψ such that

$\psi^{(m)} \longrightarrow \psi$ strongly in $L_t^\infty \mathcal{E}^{(1)}$ and <u>$*$-weakly</u> in $L_t^\infty \mathcal{E}^{(k)}$

- Recall: If X is a Banach space, then $f^{(m)} \xrightarrow{\text{weakly}} f$ $(f^{(m)}, f\in X)$

  if $\forall\, g\in X^*$, we have $g(f^{(m)}) \xrightarrow{m\to\infty} g(f)$.

  (weaker notion than strong convergence, i.e. regular convergence).

  A slightly weaker notion: <u>weak-$*$</u> convergence:

  In the special case when $X = Y^*$ for some other Banach space Y:

  $f^{(m)} \xrightarrow{\text{weak-}*} f$ iff $g(f^{(m)}) \to g(f)$ for all $g\in Y$

  (here we use the natural embedding $Y \hookrightarrow Y^{**}$)

  If X is reflexive: weak convergence = weak$*$ convergence.

Important properties:

- If $f^{(m)} \xrightarrow{\text{weakly}} f$: $\|f\| \leq \liminf_{m\to\infty} \|f^{(m)}\|$
- Banach - Alaoglu: The unit ball $\{f : \|f\| \leq 1\}$ is compact with the weak-* topology.

So in particular any bounded sequence in X has a weak-* convergent subsequence.
(this is what we use here for $L^\infty_t \mathcal{E}^{(k)}$. Note that

$$L^\infty_t \mathcal{E}^{(k)} = (L^1_t \mathcal{E}^{(-k)})^* \quad)$$

In our case: The above properties imply that ψ satisfies

$$\sup_{\tau\in[0,T]} \mathcal{E}^{(k)}[\psi](\tau) \leq \liminf_{m\to\infty} \sup_{\tau} \mathcal{E}^{(k)}[\psi^{(m)}](\tau) \leq C_0 M.$$

As a result: $\exists$ limiting function ψ such that

$\psi^{(m)} \to \psi$ strongly in $L^\infty_t H^1_x$ and weakly in $L^\infty_t H^k$.

- Recall: Weak convergence: $f^{(m)}, f \in X$ (Banach space), then $f^{(m)} \xrightarrow{\text{weakly}} f$ if $\langle g, f^{(m)} \rangle \to \langle g, f \rangle$ for all $g \in X^*$
  - If $f^{(m)} \xrightarrow{\text{weakly}} f$: $\|f\| \leq \liminf_{m\to\infty} \|f^{(m)}\|$
  - If X is a Hilbert space: The sets $\{\|\cdot\| \leq C\}$ are weakly compact.
  - In our case: We have the stronger bound

$$\sup_{\tau \in [0,T]} \mathcal{E}^{(k)}[\psi](\tau) \leq C \cdot M.$$

Moreover:

By interpolation: For any $s < k$, $\psi^{(m)}$ converges strongly to ψ in $L^\infty_t H^s_x$ (Interpolation inequality:

$$\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\frac{s_2 - s}{s_2 - s_1}} \cdot \|f\|_{H^{s_2}}^{\frac{s - s_1}{s_2 - s_1}}$$

if $s_1 < s < s_2$)

- So, since $k > 2n + 2 > \frac{n}{2} + 2$: By the Sobolev embedding, $\psi^{(m)}$ converges to ψ in C^2, so ψ solves the original equation.

In order to prove ①: We will argue inductively.

Assume it is true for all $m' \leq m-1$.

We will first establish 1^{st} order energy estimates for $\psi^{(m)}$.

Multiplying the equation with $\partial_t \psi^{(m)}$, we have:

$$G^{00}(\psi^{(m-1)})\cdot\partial_0^2\psi^{(m)}\,\partial_0\psi^{(m)} + G^{ij}(\psi^{(m-1)})\cdot\partial_{ij}^2\psi^{(m)}\cdot\partial_0\psi^{(m)}$$

$$+\,2\,G^{0i}(\psi^{(m-1)})\cdot\underbrace{\partial_0\partial_i\psi^{(m)}\cdot\partial_0\psi^{(m)}}_{\frac{1}{2}\partial_i\left((\partial_0\psi^{(m)})^2\right)} = N(\psi^{(m-1)},\partial\psi^{(m-1)})\cdot\partial\psi^{(m)}$$

So integrating by parts over $0\le t\le\tau$:

$$\int_{t=\tau} -\frac{1}{2}G^{00}(\psi^{(m-1)})\cdot(\partial_0\psi)^2 + \frac{1}{2}G^{ij}(\psi^{(m-1)})\cdot\partial_i\psi^{(m)}\partial_j\psi^{(m)}\,dx$$

$$-\int_{t=0}\cdots\cdots\cdots\,dx \qquad (A)$$

$$=\int_{0\le t\le\tau}\left[\tilde{G}(\psi^{(m-1)})\cdot\partial\psi^{(m-1)}\cdot\partial\psi^{(m)}\cdot\partial\psi^{(m)} + N(\psi^{(m-1)},\partial\psi^{(m-1)})\cdot\partial\psi^{(m)}\right]dx\,dt$$

Since $|G^{\alpha\beta}-\eta^{\alpha\beta}|<\frac{1}{10\cdot n}$, the boundary integrals are comparable with the usual energy:

$$\frac{2}{3}\mathcal{E}^{(1)}[\psi^{(m)}](\tau) \le \int_{t=\tau} -G^{00}\cdot(\partial_0\psi^{(m)})^2 + G^{ij}\,\partial_i\psi^{(m)}\cdot\partial_j\psi^{(m)}\,dx$$

$$\le \frac{3}{2}\mathcal{E}^{(1)}[\psi^{(m)}](\tau)$$

So (A) $\Longrightarrow$

$$\Rightarrow\ \mathcal{E}^{(1)}[\psi^{(m)}](\tau) \le 3\,\mathcal{E}^{(1)}[\psi^{(m)}](0) + C\underbrace{\int_{0\le t\le\tau}|\tilde{G}(\psi^{(m-1)})|\cdot|\partial\psi^{(m-1)}|\cdot|\partial\psi^{(m)}|^2 + |N(\psi^{(m-1)},\partial\psi^{(m-1)})|\cdot|\partial\psi^{(m)}|\,dx\,dt}_{(B)}$$

From the inductive step:

$$\|\psi^{(m-1)}\|_{L^\infty} + \|\partial\psi^{(m-1)}\|_{L^\infty} \underset{\text{Sobolev embedding}}{\lesssim} \sup_{\tau}\mathcal{E}[\psi^{(m-1)}]^{1/2}(\tau) \le C'\cdot M^{1/2}$$

So $\sup |\tilde{G}(\psi^{(m-1)})| \leq \sup_{z \in [-C'M^{1/2}, C'M^{1/2}]} |\tilde{G}(z)| := C_1(M)$.

and $|N(\psi^{(m-1)}, \partial\psi^{(m-1)})| \leq C_2(M) \cdot (|\psi^{(m-1)}| + |\partial\psi^{(m-1)}|)$

(since, $N(0,0)=0$ so by Taylor: $N(x,y) = N_1(x,y)\cdot x + N_2(x,y)\cdot y$)

So $$\text{B} \leq C''(M) \int_{0 \leq t \leq \tau} |\partial\psi^{(m)}|^2 + (|\psi^{(m-1)}| + |\partial\psi^{(m-1)}|)\cdot|\partial\psi^{(m)}| \, dx\, dt$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} C''(M)\cdot\left(\int_0^\tau \mathcal{E}^{(1)}[\psi^{(m)}](t)\,dt + \underbrace{\int_0^\tau \mathcal{E}^{(1)}[\psi^{(m-1)}](\tau)\,d\tau}_{\leq C\cdot M\cdot T(M)}\right)$$

So: $$\mathcal{E}^{(1)}[\psi^{(m)}](\tau) \leq 3\,\mathcal{E}^{(1)}[\psi^{(m)}](0) + C''(M)\cdot M\cdot T + C''(M)\int_0^\tau \mathcal{E}^{(1)}[\psi^{(m)}](t)\,dt$$

By Gronwall: $$\mathcal{E}^{(1)}[\psi^{(m)}](\tau) \leq e^{C'''(M)\cdot T(M)}\left(3\,\mathcal{E}^{(1)}[\psi^{(m)}](0) + C''(M)\cdot T(M)\right)$$

independent of M!

So if $T(M)$ is small enough in terms of $C'''(M)$:

$$\mathcal{E}^{(1)}[\psi^{(m)}](\tau) \leq C_0\, M$$

the original constant!

I need to obtain the same bound for $\mathcal{E}^{(k)}[\psi^{(m)}]$!

Commute with ∂^a, $|a| \leq k-1$, repeat the same steps: (higher order energy estimate)

$$\mathcal{E}^{(k)}[\psi^{(m)}](\tau) \leq 3\cdot\mathcal{E}^{(k)}[\psi^{(m)}](0) + C\int_{0\leq t\leq \tau} \tilde{F}(\partial^{\leq \frac{k}{2}+1}\psi^{(m-1)})\Big(|\partial^{\leq k}\psi^{(m-1)}|^2 + |\partial^{\leq k}\psi^{(m)}|^2\Big)\,dx\,dt$$

term with the fewest number of derivatives

Since $\|\partial^{\leq \frac{k}{2}+1} f\|_{L^\infty} \lesssim \|f\|_{H^k}$ if $k > 2n+2$:

By using as before the inductive assumption & Gronwall, we get if $T(M)$ is chosen small in term of M:

$$\mathcal{E}^{(k)}[\psi^{(m)}] \leq C_0\cdot M.$$

For the difference estimate:

Let $v^{(m)} = \psi^{(m)} - \psi^{(m-1)}$. Subtracting the equation for $\psi^{(m-1)}$ from that for $\psi^{(m)}$:

$$\partial_\alpha\big(G^{\alpha\beta}(\psi^{(m-1)})\cdot\partial_\beta(\underbrace{\psi^{(m)}-\psi^{(m-1)}}_{v^{(m)}})\big) + \partial_\alpha\big([G^{\alpha\beta}(\psi^{(m-1)}) - G^{\alpha\beta}(\psi^{(m-2)})]\partial_\beta\psi^{(m-1)}\big)$$
$$= N(\psi^{(m-1)}, \partial\psi^{(m-1)}) - N(\psi^{(m-2)}, \partial\psi^{(m-1)})$$

We can express this as a linear equation in $v^{(m)}$:

- $$G(\psi^{(m-1)}) - G(\psi^{(m-2)}) = \int_0^1 \frac{d}{d\lambda} G((1-\lambda)\cdot\psi^{(m-2)} + \lambda\psi^{(m-1)})\,d\lambda$$
$$= \underbrace{\Big(\int_0^1 G'((1-\lambda)\psi^{(m-2)} + \lambda\cdot\psi^{(m-1)})\,d\lambda\Big)}_{=:\tilde{G}(\psi^{(m-1)},\,\psi^{(m-2)})}\cdot(\psi^{(m-1)} - \psi^{(m-2)})$$

- Similarly: $N(\psi^{(m-1)}, \partial\psi^{(m-1)}) - N(\psi^{(m-2)}, \partial\psi^{(m-1)}) =$

$$= \tilde{N}_1(\psi^{(m-2)}, \partial\psi^{(m-2)}, \psi^{(m-1)}, \partial\psi^{(m-1)}) \cdot (\psi^{(m-1)} - \psi^{(m-2)})$$
$$+ \tilde{N}_2(\quad \cdots \quad) \cdot (\partial\psi^{(m-1)} - \partial\psi^{(m-2)}).$$

So $v^{(m)}$ satisfies

$$G^{\alpha\beta}(\psi^{(m-1)}) \cdot \partial_\alpha \partial_\beta v^{(m)} + F_1(\partial^{\leq 1}\psi^{(m-2)}, \partial^{\leq 1}\psi^{(m-1)}) \cdot \partial v^{(m)}$$
$$+ F_2(\partial^{\leq 1}\psi^{(m-2)}, \partial^{\leq 1}\psi^{(m-1)}) \cdot v^{(m)} =$$
$$= H_1(\partial^{\leq 2}\psi^{(m-2)}, \partial^{\leq 2}\psi^{(m-1)}, \underset{\text{also } \psi^{(m)}}{\uparrow}) \cdot \partial v^{(m-1)}$$
$$+ H_2(\quad \cdots \quad) \cdot v^{(m-1)}$$

With initial data

$$v^{(m)}|_{t=0} = 0, \quad \partial_t v^{(m)}|_{t=0} = 0.$$

So energy estimate:

$$\mathcal{E}^{(1)}[v^{(m)}](\tau) \leq 3\, \underbrace{\mathcal{E}^{(1)}[v^{(m)}](0)}_{=0} + C\, \|\partial^{\leq 2}\psi^{(m-2, m-1, m)}\|_{L^\infty} \int_0^\tau \mathcal{E}^{(1)}[v^{(m-1)}] + \mathcal{E}^{(1)}[v^{(m)}]$$

Since by Sobolev (and the energy estimate in the last step)

we have $\|\partial^{\leq 2}\psi^{(m-2, m-1, m)}\|_{L^\infty} \leq C \cdot M^{1/2}$, applying Gronwall

we get
$$\sup_\tau \mathcal{E}^{(1)}[v^{(m)}](\tau) \leq C' e^{C(M)T(M)} \cdot M^{1/2} \cdot T(M) \cdot \sup_\tau \mathcal{E}^{(1)}[v^{(m-1)}](\tau)$$

So if $T(M)$ small enough in terms of M:

The constant $\leq \frac{1}{2}$. $\square$

Uniqueness / continuous dependence:

Apply the energy estimate for the difference of two solutions. $\square$

Remark:

- The proof would work for quasilinear equations with symbol
$$G^{\alpha\beta}(\psi, \partial\psi)$$

- If $N(0,0) = 0$, then $\psi = 0$ is the solution with initial data $(\psi_0, \psi_1) = (0,0)$.

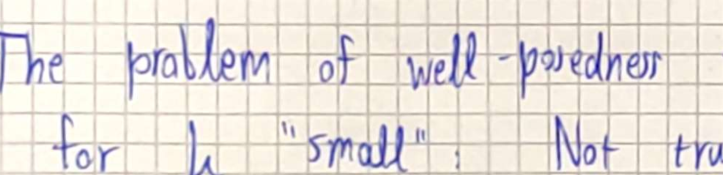
Continuous dependence on initial data: As $(\psi_0, \psi_1) \xrightarrow{H^k \times H^{k-1}} (0,0)$ then:

  - $T \longrightarrow +\infty$
  - $\forall\, T_0 > 0$: on $[0, T_0] \times \mathbb{R}^n$, $\|\psi\|_{L^\infty_t H^k_x} + \|\partial_t \psi\|_{L^\infty_t H^{k-1}} \longrightarrow 0$.

(so uniform convergence over compact domains)

- The problem of well-posedness for initial data in $H^k \times H^{k-1}$ for k "small": Not true if k is too small

- For more general equations:



Work in coordinates where g is close to η, Glue the local solutions using uniqueness